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Compactly Generated de Morgan Lattices, Basic Algebras and Effect Algebras

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Abstract We prove that a de Morgan lattice is compactly generated if and only if its order topology is compatible with a uniformity on L generated by some separating function family on L. Moreover, if L is complete then L is (o)-topological. Further, if a basic algebra L (hence lattice with sectional antitone involutions) is compactly generated then L is atomic. Thus all non-atomic Boolean algebras as well as non-atomic lattice effect algebras (including non-atomic MV-algebras and orthomodular lattices) are not compactly generated.

Keywords Compact element · Compactly generated · de Morgan lattice · Basic algebra · Lattice effect algebra

1 Introduction

In 1992, in the study of axiomatic system of fuzzy sets Kôpka [8] defined a new structure, a so called D-poset of fuzzy sets, which is closed under the formation of differences of fuzzy sets. A generalization of a D-poset of fuzzy sets to an abstract partially ordered set, where the basic operation is the difference were introduced in [9]. Simultaneously it was introduced an equivalent in some sense structure called effect algebra [4] as a generalization of Hilbert-space effects interpreted as the "unsharp" quantum events. Differently from the "sharp" events the effects do not satisfy the non-contradiction principle, i.e., the conjunction of a and non a may be different from zero. These new logical structures for presence of propositions, properties, questions or events with fuzziness, uncertainty or unsharpness

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generalize orthomodular lattices (including Boolean algebras) as well as MV-algebras employed by C.C. Chang in the analysis of many valued logics [2]. E.g., carriers of probabilities in quantum or fuzzy theory are effect algebras.

A partial algebra $(E; \oplus, 0, 1)$ is called an *effect algebra* if 0, 1 are two distinct elements and $\oplus$ is a partially defined binary operation on E which satisfy the following conditions for any $a, b, c \in E$:

(Ei) $b \oplus a = a \oplus b$ if $a \oplus b$ is defined,
(Eii) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$ if one side is defined,
(Eiii) for every $a \in E$ there exists a unique $b \in E$ such that $a \oplus b = 1$ (we put $a' = b$),
(Eiv) if $1 \oplus a$ is defined then $a = 0$.

We often denote the effect algebra $(E; \oplus, 0, 1)$ briefly by E. In every effect algebra E we can define the partial order $\leq$ by putting $a \leq b$ and $b \ominus a = c$ iff $a \oplus c$ is defined and $a \oplus c = b$.

If E with the defined partial order is a lattice (a complete lattice) then $(E; \oplus, 0, 1)$ is called a *lattice effect algebra* (*a complete lattice effect algebra*).

Recall that a set $Q \subseteq E$ is called a sub-effect algebra of the effect algebra E if

(i) $1 \in Q$
(ii) if out of elements $a, b, c \in E$ with $a \oplus b = c$ two are in Q, then $a, b, c \in Q$.

If a sub-effect algebra Q of a lattice effect algebra E is simultaneously a sublattice of E then Q is called *sub-lattice effect algebra*.

We say that a finite system $F = (a_k)_{k=1}^n$ of not necessarily different elements of an effect algebra $(E; \oplus, 0, 1)$ is $\oplus$-orthogonal if $a_1 \oplus a_2 \oplus \cdots \oplus a_n$ (written $\bigoplus_{k=1}^n a_k$ or $\bigoplus F$) exists in E. Here we define $a_1 \oplus a_2 \oplus \cdots \oplus a_n = (a_1 \oplus a_2 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ supposing that $\bigoplus_{k=1}^{n-1} a_k$ exists and $\bigoplus_{k=1}^{n-1} a_k \leq a_n'$. An arbitrary system $G = (a_\kappa)_{\kappa \in H}$ of not necessarily different elements of E is $\oplus$-orthogonal if $\bigoplus K$ exists for every finite $K \subseteq G$. We say that for a $\oplus$-orthogonal system $G = (a_\kappa)_{\kappa \in H}$ the element $\bigoplus G$ exists iff $\bigvee\{\bigoplus K \mid K \subseteq G, K$ is finite$\}$ exists in E and then we put $\bigoplus G = \bigvee\{\bigoplus K \mid K \subseteq G\}$ (we write $G_1 \subseteq G$ iff there is $H_1 \subseteq H$ such that $G_1 = (a_\kappa)_{\kappa \in H_1}$).

Finally note that lattice effect algebras generalize orthomodular lattices [6] (including Boolean algebras) if we assume existence of unsharp elements $x \in E$, meaning that $x \wedge x' \neq 0$. On the other hand the *set* $S(E) = \{x \in E \mid x \wedge x' = 0\}$ *of all sharp elements* of a lattice effect algebra E is an orthomodular lattice [5]. In this sense a lattice effect algebra is a “smeared” orthomodular lattice. An orthomodular lattice L can be organized into a lattice effect algebra by setting $a \oplus b = a \vee b$ for every pair $a, b \in L$ such that $a \leq b^\perp$.

For an element x of an effect algebra E we write $\mathrm{ord}(x) = \infty$ if $nx = x \oplus x \oplus \cdots \oplus x$ (n-times) exists for every positive integer n and we write $\mathrm{ord}(x) = n_x$ if n_x is the greatest positive integer such that $n_x x$ exists in E. An effect algebra E is *Archimedean* if $\mathrm{ord}(x) < \infty$ for all $x \in E$. We can show that every complete effect algebra is Archimedean (see [11]). An element a of an effect algebra E is an *atom* if $0 \leq b < a$ implies $b = 0$ and E is called *atomic* if for every nonzero element $x \in E$ there is an atom a of E with $a \leq x$. An element $u \in E$ is called *finite* if either $u = 0$ or there is a finite sequence $\{p_1, p_2, \ldots, p_n\}$ of not necessarily different atoms of E such that $u = p_1 \oplus p_2 \oplus \cdots \oplus p_n$.

The aim of this paper is to study compactly generated lattice effect algebras (or more general compactly generated basic algebras). Our main result states that all of them must be atomic. Moreover, we show that a de Morgan lattice is compactly generated if and only if its order topology is compatible with a uniformity on L generated by some separating function family on L.

2 Compactly Generated de Morgan Lattices

Definition 1 A structure $(E \leq,', 0, 1)$ is called a *de Morgan lattice* if $(E, \leq)$ is a lattice and $'$ is a unary operation with properties $a \leq b \Rightarrow b' \leq a'$ and $a = a''$. We will speak sometimes about a *duality operation* $'$.

In fact in a de Morgan lattice we have $a \leq b$ iff $b' \leq a'$, because $a \leq b \Rightarrow b' \leq a' \Rightarrow a'' \leq b'' \Rightarrow a \leq b$. If $(E, \oplus, 0, 1)$ is an effect algebra, then for $a \in E$ there is a unique $a' \in E$ such that $a \oplus a' = 1$. Moreover, E is a poset with $a \leq c$ iff $a \oplus b$ is defined for some $b \in E$ such that $a \oplus b = c$. It easy to see that then a lattice effect algebra $(E, \leq,')$ is a de Morgan lattice [10].

Definition 2 A net $(a_\alpha)_{\alpha\in\mathcal{E}}$ of elements of a poset $(P; \leq)$ *order converges* ((o)*-converges*, for short) to a point $a \in P$ if there are nets $(u_\alpha)_{\alpha\in\mathcal{E}}$ and $(v_\alpha)_{\alpha\in\mathcal{E}}$ of elements of P such that

$$a \uparrow u_\alpha \leq a_\alpha \leq v_\alpha \downarrow a\,.$$

We write $a_\alpha \stackrel{(o)}{\to} a$. The *order topology* τ_o on P is the finest topology on P such that for every net $(a_\alpha)_{\alpha\in\mathcal{E}}$ of elements of P $a_\alpha \stackrel{(o)}{\to} a$ in $P \Rightarrow a_\alpha \stackrel{\tau_o}{\to} a$, where $a_\alpha \stackrel{\tau_o}{\to} a$ denotes that $(a_\alpha)_{\alpha\in\mathcal{E}}$ *converges* to $a \in P$ in the topological space (P, τ_o).

Lemma 1 *Let E be a de Morgan lattice. Then $a_\alpha \uparrow a$ iff $a'_\alpha \downarrow a'$ and $a_\alpha \stackrel{(o)}{\to} a$ iff $a'_\alpha \stackrel{(o)}{\to} a'$.*

Proof Let $a_\alpha \uparrow a$. Then for all $\alpha_1, \alpha_2 \in \mathcal{E}$, $\alpha_1 \leq \alpha_2$ we have $a_{\alpha_1} \leq a_{\alpha_2}$ hence $a'_{\alpha_2} \leq a'_{\alpha_1}$, which implies that $a'_\alpha \downarrow$. Now $a_\alpha \uparrow a$ means that $a = \bigvee\{a_\alpha | \alpha \in \mathcal{E}\}$ and hence $a_\alpha \leq a$ for all $\alpha \in \mathcal{E}$. This implies $a' \leq a'_\alpha$. Let $d \leq a'_\alpha$ for all $\alpha \in \mathcal{E}$ then $a_\alpha \leq d'$ for all $\alpha \in \mathcal{E}$ which implies $a \leq d'$ hence $d \leq a'$. This proves that $a'_\alpha \downarrow a'$. Now the rest is obvious. □

Remark 1 It is easy to check that, for an (o)-continuous de Morgan lattice E and for any x_α, $y_\alpha, x, y \in E$, we have $x_\alpha \xrightarrow{(o)} x$, $y_\alpha \xrightarrow{(o)} y$ implies $x_\alpha \vee y_\alpha \xrightarrow{(o)} x \vee y$ and $x_\alpha \wedge y_\alpha \xrightarrow{(o)} x \wedge y$.

Definition 3 A subset $\mathcal{U}$ of a bounded lattice $L = (L, \vee, \wedge, 0, 1)$ is *join-dense* if for any two elements $x, z \in L$ with $x \not\leq z$, there is some $u \in \mathcal{U}$ with $u \leq x$ but $u \not\leq z$. Thus $\mathcal{U}$ is join-dense in L iff each element of L is a join of elements from $\mathcal{U}$. *Meet-density* is defined dually.

Let L be a lattice such that there exist $\mathcal{U}, \mathcal{V} \subseteq L$ such that for every $x \in L$ we have that

$$x = \bigvee\{u \in \mathcal{U} \mid u \leq x\} = \bigwedge\{v \in \mathcal{V} \mid x \leq v\}.$$

Consider the function family $\Phi = \{f_u \mid u \in \mathcal{U}\} \cup \{g_v \mid v \in \mathcal{V}\}$, where $f_u, g_v : L \to \{0, 1\}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ are defined by putting

$$f_u(x) = \begin{cases} 1 & \text{iff } u \leq x, \\ 0 & \text{iff } u \not\leq x \end{cases} \quad \text{and} \quad g_v(y) = \begin{cases} 1 & \text{iff } x \leq v, \\ 0 & \text{iff } x \not\leq v, \end{cases} \quad \text{for all } x, y \in L.$$

Further, consider the family of pseudometrics on L: $\Sigma_\Phi = \{\rho_u \mid u \in \mathcal{U}\} \cup \{\pi_v \mid v \in \mathcal{V}\}$, where $\rho_u(a, b) = |f_u(a) - f_u(b)|$ and $\pi_v(a, b) = |g_v(a) - g_v(b)|$ for all $a, b \in L$.

Let us denote by $\mathcal{U}_\Phi$ the uniformity on L induced by the family of pseudometrics Σ_Φ (see e.g. [3]). Further denote by τ_Φ the topology compatible with the uniformity $\mathcal{U}_\Phi$.

Then for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of L

$$x_\alpha \xrightarrow{\tau_\Phi} x \quad \text{iff } \varphi(x_\alpha) \to \varphi(x) \quad \text{for any } \varphi \in \Phi.$$

This implies, since f_u, $u \in \mathcal{U}$, and g_v, $v \in \mathcal{V}$, is a separating function family on L, that the topology τ_Φ is Hausdorff. Moreover, the intervals $[u, v] = [u, 1] \cap [0, v] = f_u^{-1}(\{1\}) \cap g_v^{-1}(\{1\})$ are clopen sets in τ_Φ. Hence any interval $[\bigvee_{i=1}^n u_i, \bigwedge_{i=1}^n v] = \bigcap_{i=1}^n [u_i, v_i]$, $u_i \in \mathcal{U}$, $v_i \in \mathcal{V}$ is clopen in τ_Φ.

Recall that a subset $\mathcal{W} \subseteq L$ is called *directed* if for any two elements $x, y \in \mathcal{W}$ there is an element $z \in \mathcal{W}$ such that $x \leq z$ and $y \leq z$. Filtered subsets are defined dually.

Lemma 2 *Let L be a de Morgan lattice such that there exist $\mathcal{U}, \mathcal{V} \subseteq L$, $\mathcal{U}$ directed and join-dense in L and $\mathcal{V}$ filtered and meet-dense in L. Then $\tau_o \subseteq \tau_\Phi$.*

Proof Let $x \in L$ be arbitrary and $x \in U \in \tau_o$. Put $P_x = \{u \in L \mid u \leq x,\ u \in \mathcal{U}\}$ and $Q_x = \{v \in L \mid x \leq v,\ v \in \mathcal{V}\}$. Then $x = \bigvee P_x$ and $x = \bigwedge Q_x$. For every finite set $F \subseteq P_x \cup Q_x$ we put $u_F = \bigvee(F \cap P_x)$ and $v_F = \bigwedge(F \cap Q_x)$. Evidently $\mathcal{E} = \{F \subseteq P_x \cup Q_x \mid F \text{ is finite}\}$ is directed by set inclusion and $u_F \uparrow x$, $v_F \downarrow x$. Since $x \in U \in \tau_o$ there is $F_0 \in \mathcal{E}$ such that $[u_{F_0}, v_{F_0}] \subseteq U$ (see, e.g., Appendix B by H. Kirchheimová and Z. Riečanová, Proposition B.2.1 (ii) in [7]) and the interval $[u_{F_0}, v_{F_0}]$ is τ_Φ-clopen by the above remark. Hence any open set from τ_o is a union of clopen sets from τ_Φ. □

Definition 4 (1) An element a of a lattice L is called *compact* iff, for any $D \subseteq L$ with $\bigvee D \in L$, if $a \leq \bigvee D$ then $a \leq \bigvee F$ for some finite $F \subseteq D$.

(2) A lattice L is called *compactly generated* iff every element of L is a join of compact elements.

The notions of *cocompact element* and *cocompactly generated lattice* can be defined dually.

Theorem 1 *Let L be a de Morgan lattice such that there exist $\mathcal{U}, \mathcal{V} \subseteq L$, $\mathcal{U}$ directed and join-dense in L and $\mathcal{V}$ filtered and meet-dense in L. Then the following conditions are equivalent*:

1. $\tau_o = \tau_\Phi$.
2. *Elements of $\mathcal{U}$ are compact and elements of $\mathcal{V}$ are cocompact. Hence L is compactly generated by $\mathcal{U}$.*

Proof (1) $\Longrightarrow$ (2): Let $u \in \mathcal{U}$, $D \subseteq L$, $u \leq \bigvee D = x$. For every finite $F \subseteq D$ we set $d_F = \bigvee F$. Then $d_F \uparrow x$. Hence $d_F \xrightarrow{(o)} x$ and therefore $d_F \xrightarrow{\tau_o} x$. Since $\tau_o = \tau_\Phi$ we have that $d_F \xrightarrow{\tau_\Phi} x$. Thus $f_u(d_F) \to f_u(x)$. That means that there exists a finite subset $F_0 \subseteq D$ such that for all finite $F \subseteq D$, $F_0 \subseteq F$ we have that $1 = f_u(x) = f_u(d_F)$. Therefore $u \leq d_{F_0}$. The case $v \in \mathcal{V}$ can be proved dually.

(2) $\Longrightarrow$ (1): Let us show that $x_\alpha \xrightarrow{(o)} x$ implies $x_\alpha \xrightarrow{\tau_\Phi} x$. Assume that $u_\alpha \leq x_\alpha \leq v_\alpha$ for all α such that $u_\alpha \uparrow x$ and $v_\alpha \downarrow x$. Let $u \in \mathcal{U}$. If $f_u(x) = 0$ we have that $u \not\leq x$. Therefore $u \not\leq u_\alpha$ for all α i.e. $f_u(u_\alpha) = 0$. Moreover there exists an index α_0 such that $u \not\leq v_{\alpha_0}$ i.e. $f_u(v_\alpha) = 0$ for all $\alpha \geq \alpha_0$. If $f_u(x) = 1$ we have that $u \leq x$. Hence by compactness of u

there is an index α_0 such that $u \le u_{\alpha_0}$. This immediately implies that for all $\alpha \ge \alpha_0$ we have $u \le x_\alpha$ i.e. $f_u(u_\alpha) = 1$. Clearly, $u \le v_\alpha$ for all α i.e. $f_u(v_\alpha) = 1$.

Hence in both cases we have that $f_u(x_\alpha)$ is eventually constant. Therefore $f_u(x_\alpha) \to f_u(x)$. The case $v \in \mathcal{V}$ can be proved dually. Hence we have, for all $u \in \mathcal{U}$ and for all $v \in \mathcal{V}$, $f_u(x_\alpha) \to f_u(x)$ and $g_v(x_\alpha) \to g_v(x)$.

Since τ_o is the finest (strongest) topology on L such that $x_\alpha \xrightarrow{(o)} x$ implies topological convergence we have that $\tau_o \supseteq \tau_\Phi$. The reverse inclusion follows from Lemma 2. □

Lemma 3 *Let L be a complete de Morgan lattice such that there exist $\mathcal{U}, \mathcal{V} \subseteq L$, $\mathcal{U}$ directed and join-dense in L and $\mathcal{V}$ filtered and meet-dense in L. Then, for any net (x_α) of L and any $x \in L$, $x_\alpha \xrightarrow{\tau_\Phi} x$ implies $x_\alpha \xrightarrow{(o)} x$.*

Proof Let $x_\alpha \xrightarrow{\tau_\Phi} x$. It is easy to check that $x_\alpha \xrightarrow{(o)} x$ iff $x = \bigvee_{\beta\in\mathcal{E}} \bigwedge_{\alpha\ge\beta} x_\alpha = \bigwedge_{\beta\in\mathcal{E}} \bigvee_{\alpha\ge\beta} x_\alpha$. For every $u \in \mathcal{U}$, $u \le x$ there exists an index $\beta(u)$ such that for every $\alpha \ge \beta(u)$ we have $u \le x_\alpha$. Therefore $u \le x$, $u \in \mathcal{U}$ implies $u \le \bigvee_{\beta\in\mathcal{E}} \bigwedge_{\alpha\ge\beta} x_\alpha$ and hence $x \le \bigvee_{\beta\in\mathcal{E}} \bigwedge_{\alpha\ge\beta} x_\alpha$. The case $v \in \mathcal{V}$ can be proved dually. Hence we have that $x \ge \bigwedge_{\beta\in\mathcal{E}} \bigvee_{\alpha\ge\beta} x_\alpha$. Now $x \le \bigvee_{\beta\in\mathcal{E}} \bigwedge_{\alpha\ge\beta} x_\alpha \le \bigwedge_{\beta\in\mathcal{E}} \bigvee_{\alpha\ge\beta} x_\alpha \le x$. This means that $x_\alpha \xrightarrow{(o)} x$. □

Theorem 2 *Let L be a complete de Morgan lattice such that there exist $\mathcal{U}, \mathcal{V} \subseteq L$, $\mathcal{U}$ directed and join-dense in L and $\mathcal{V}$ filtered and meet-dense in L. Then the following conditions are equivalent*:

1. *For any net (x_α) of L and any $x \in L$, $x_\alpha \xrightarrow{\tau_\Phi} x$ if and only if $x_\alpha \xrightarrow{(o)} x$.*
2. *L is compactly generated by $\mathcal{U}$ and cocompactly generated by $\mathcal{V}$.*

Moreover, any of the conditions (1) *and* (2) *implies the condition* (3).

3. *L is (o)-topological i.e., order convergence of nets of elements coincides with convergence in the order topology τ_o and it makes lattice and duality operations continuous.*

Proof (2) $\Longrightarrow$ (1): From the proof of Theorem 1 we know that $x_\alpha \xrightarrow{(o)} x$ implies $x_\alpha \xrightarrow{\tau_\Phi} x$. The reverse implication follows from Lemma 3.
(1) $\Longrightarrow$ (2): This follows evidently from the proof of Theorem 1.
(1) $\Longrightarrow$ (3): In view of Theorem 1 we have that $\tau_o = \tau_\Phi$ and L is o-continuous. From Lemma 1 and Remark 1 we have that L is (o)-topological with respect to the topology τ_Φ. □

Let L be a de Morgan lattice such that such that there exist $\mathcal{U}, \mathcal{V} \subseteq L$, $\mathcal{U}$ join-dense in L and $\mathcal{V}$ meet-dense in L. Let $MC(L) = \widehat{L}$ be the MacNeille completion of L.

Then for every $x \in \widehat{L}$ we have that $x = \bigvee\{u \in \mathcal{U} \mid u \le x\} = \bigwedge\{v \in \mathcal{V} \mid x \le v\}$.

Consider the function family $\widehat{\Phi} = \{\widehat{f_u} \mid u \in \mathcal{U}\} \cup \{\widehat{g_v} \mid v \in \mathcal{V}\}$, $\widehat{f_u}, \widehat{g_v} : \widehat{L} \to \{0, 1\}$, $u \in \mathcal{U}, v \in \mathcal{V}$ are defined by putting

$$f_u(x) = \begin{cases} 1 & \text{iff } u \le x \\ 0 & \text{iff } u \not\le x \end{cases}, \qquad g_v(y) = \begin{cases} 1 & \text{iff } x \le v \\ 0 & \text{iff } x \not\le v \end{cases}$$

for all $x, y \in \widehat{L}$.

Let us consider the family of pseudometrics on $\widehat{L}$: $\Sigma_{\widehat{\Phi}} = \{\widehat{\rho_u} \mid u \in \mathcal{U}\} \cup \{\widehat{\pi_v} \mid v \in \mathcal{V}\}$, where $\widehat{\rho_u}(a, b) = |\widehat{f_u}(a) - \widehat{f_u}(b)|$ and $\widehat{\pi_v}(a, b) = |\widehat{g_v}(a) - \widehat{g_v}(b)|$ for all $a, b \in \widehat{L}$.

We will denote by $\mathcal{U}_{\widehat{\Phi}}$ the uniformity on $\widehat{L}$ induced by the family of pseudometrics $\Sigma_{\widehat{\Phi}}$ by $\widehat{\tau}_{\widehat{\Phi}}$ the topology compatible with the uniformity $\mathcal{U}_{\widehat{\Phi}}$. We have, for every net $(x_\alpha)_{\alpha\in\mathcal{E}}$ of elements of $\widehat{L}$

$$x_\alpha \xrightarrow{\widehat{\tau}_{\widehat{\Phi}}} x \quad \text{iff } \widehat{\varphi}(x_\alpha) \to \widehat{\varphi}(x) \quad \text{for any } \widehat{\varphi} \in \widehat{\Phi}.$$

The Lemma 2 ensures that, for $\mathcal{U}$ directed and $\mathcal{V}$ filtered, $\widehat{\tau}_o \subseteq \widehat{\tau}_{\widehat{\Phi}}$. We obtain the following common corollary of Theorems 1 and 2.

Theorem 3 *Let L be a de Morgan lattice such that there exist $\mathcal{U}, \mathcal{V} \subseteq L$, $\mathcal{U}$ directed and join-dense in L and $\mathcal{V}$ filtered and meet-dense in L. Then the following conditions are equivalent*:

1. $\widehat{\tau}_o = \widehat{\tau}_{\widehat{\Phi}}$.
2. *Elements of $\mathcal{U}$ are compact in $\widehat{L}$ and elements of $\mathcal{V}$ are cocompact in $\widehat{L}$. Hence $\widehat{L}$ is compactly generated by $\mathcal{U}$.*
3. *For any net (x_α) of $\widehat{L}$ and any $x \in \widehat{L}$, $x_\alpha \xrightarrow{\widehat{\tau}_{\widehat{\Phi}}} x$ if and only if $x_\alpha \xrightarrow{\widehat{(o)}} x$.*

By ([15], Theorem 5.4 and 5.5) every complete atomic modular lattice effect algebra E is compactly generated by finite elements and E is (o)-continuous. Thus Theorem 2 and [7, Proposition 2.1] have the following corollary:

Theorem 4 *Let E be a complete atomic modular lattice effect algebra. Then*

1. *E is an (o)-topological lattice.*
2. *τ_o has a base of clopen sets of the form $[u, v]$, where u, v' are finite elements in E.*

Note that examples of modular lattice effect algebras are all MV-algebras (lattice effect algebras with a unique block) because they are distributive lattices. Moreover, modular are all lattice effect algebras possessing (o)-continuous faithful valuations (see [13], Theorem 5.4).

3 Compactly Generated Basic Algebras

A **basic algebra** [1] (**lattice with sectional antitone involutions**) is a system $\mathbf{L} = (L; \vee, \wedge, ({}^a)_{a\in L}, 0, 1)$, where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice such that every principal order-filter $[a, 1]$ (which is called a **section**) possesses an antitone involution $x \mapsto x^a$.

Clearly, any principal ideal $[0, x]$, $x \in L$ of a basic algebra L is again a basic algebra.

Theorem 5 *Let L be a compactly generated basic algebra. Then L is atomic.*

Proof Let $x \in L$, $x \neq 0$. Since L is compactly generated there is a compact element $u \in L$, $u \neq 0$ such that $u \leq x$. We have a duality, say $^\sharp$ on $[0, u]$. By Hausdorff's maximality principle there is a maximal chain C in the sub-poset $(0, u] = \{y \in L : 0 < y \leq u\}$.

Either 0 is the unique lower bound of C in the poset $[0, u]$ or there is some non-zero lower bound of C.

(a) If 0 is the unique lower bound of C in the poset $[0, u]$ then $u = 0^\sharp$ is the unique upper bound of the chain $C^\sharp = \{c^\sharp \mid c \in C\}$ and hence $u = \bigvee C^\sharp$. Because u is compact there

exist $c_1^\sharp, \dots, c_m^\sharp \in C^\sharp$ such that $u \le c_1^\sharp \vee \cdots \vee c_m^\sharp \le \bigvee C^\sharp = u$. It follows that there exists an element $c_0^\sharp \in C^\sharp \subseteq [0, u]$ such that $c_1^\sharp, \dots, c_m^\sharp \le c_0^\sharp$. Hence $u \le c_0^\sharp \le u$ which implies that $c_0 = c_0^{\sharp\sharp} = u^\sharp = 0$, a contradiction with $C \subseteq (0, u]$.
(b) Let a be a non-zero lower bound of C. Then $a \in (0, u]$ and $a \le c$ for all $c \in C$. Clearly, $C \cup \{a\}$ is again a chain in $(0, u]$. Since $C \subseteq C \cup \{a\}$ and C is a maximal chain in $(0, u]$ we obtain that $a \in C$. Hence a is the unique non-zero lower bound of C and therefore an atom of $[0, u]$. This yields that a is also an atom of L such that $a \le u \le x$. □

A well known example of a basic algebra is a lattice effect algebra $(E; \oplus, 0, 1)$. Really, if $x \in E$, $x \ne 0$ then for every $y \in [0, x]$ there exists the unique $y^\sharp$ such that $y \oplus y^\sharp = x$, hence $y^\sharp = x \ominus y$. Evidently $y_1 \le y_2$ implies $y_2^\sharp \le y_1^\sharp$ and $y^{\sharp\sharp} = y$ for all $y, y_1, y_2 \in [0, x]$. Thus we obtain the following corollary of Theorem 5.

Theorem 6 *Let $(E; \oplus, 0, 1)$ be a compactly generated lattice effect algebra. Then E is atomic.*

Lemma 4 *Let $(E; \oplus, 0, 1)$ be an atomic Archimedean lattice effect algebra. Then every compact element of E is finite.*

Proof Assume that $u \in E$ is compact. Either $u = 0$ and we are finished or $u \ne 0$. Then by Theorem [14, Theorem 3.3] there exist a set of atoms $\{a_\alpha \mid \alpha \in \mathcal{A}\}$ and positive integers $k_\alpha, \alpha \in \mathcal{A}$ such that

$$u = \bigoplus \{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\} = \bigvee \{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\}.$$

Since u is compact there exists a finite set $\mathcal{A}_0 \subseteq \mathcal{A}$ such that $u \le \bigvee \{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}_0\} = \bigoplus \{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}_0\} \le \bigoplus \{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}\} = u$. Hence $u = \bigoplus \{k_\alpha a_\alpha \mid \alpha \in \mathcal{A}_0\}$ is finite. □

The next example shows that in Lemma 4 the assumption that E is Archimedean cannot be omitted.

Example 1 Assume the so-called Chang MV-effect algebra $E = \{0, a, 2a, \dots, (2a)', a, 1\}$, i.e., ka, $(ka)' \in E$ for every positive integer k. Then evidently E is compactly generated lattice since every element of E is compact. Nevertheless, elements $(ka)'$ for every positive integer k are not finite.

Clearly E is a non-Archimedean atomic lattice effect algebra, as $\operatorname{ord}(a) = \infty$. Thus E is not compactly generated by finite elements. Finally note that there is no complete lattice effect algebra with E being its sub-lattice effect algebra, since every complete lattice effect algebra is Archimedean (see [11]).

Note that by Theorem 6 we obtain that e.g., similarly to a non-atomic complete Boolean algebra B also the standard MV-effect algebra $M = [0, 1] \subseteq \mathbb{R}$ of real numbers is not compactly generated in spite of that they both are (o)-continuous lattices. The next theorem gives necessary and sufficient conditions for that.

Theorem 7 *Let E be an Archimedean lattice effect algebra. Then the following conditions are equivalent*:

(i) *E is a compactly generated lattice.*

(ii) *E is an atomic (o)-continuous lattices.*
(iii) *E is a compactly generated lattice by finite elements.*

Proof (i) $\Longrightarrow$ (ii): By Theorem 6 E is atomic and evidently every compactly generated lattice is (o)-continuous.

(ii) $\Longrightarrow$ (iii): Let $D \subseteq E$ and $x = \bigvee D$ exist in E. Let $u = a_1 \oplus a_2 \oplus \cdots \oplus a_n$, where $a_k \in E$, for $k = 1, \ldots, n$ are not necessarily different atoms. Set $\mathcal{F} = \{F \mid F \subseteq D,\ F$ is finite$\}$. Then $\mathcal{F}$ is a directed set by partial order $F_1 \leq F_2$ iff $F_1 \subseteq F_2$. For every $F \in \mathcal{F}$ we put $x_F = \bigvee F$. Then $x_F \uparrow \bigvee D = x$.

If $u \leq x$ then by (o)-continuity of E we obtain that $a_1 \wedge x_F \uparrow a_1 \wedge x = a_1$. Because $a_1 \wedge x_F$ is equal to a_1 or 0, there is $F_1 \in \mathcal{F}$ such that for all $F \geq F_1$ we have $a_1 \wedge x_F = a_1 \leq x_F$. This implies that $x_F \ominus a_1 \uparrow x \ominus a_1$, for $F \geq F_1$. In the same manner there is $F_2 \geq F_1$, $F_2 \in \mathcal{F}$ such that $((x_F) \ominus a_1) \ominus a_2 \uparrow (x \ominus a_1) \ominus a_2$, $a_1 \oplus a_2 \leq x_{F_2} \leq x_F$ for $F \geq F_2$. By induction there is $F_n \in \mathcal{F}$ such that $a_1 \oplus a_2 \oplus \cdots \oplus a_n \leq x_{F_n} \leq x_F$ for $F \in \mathcal{F}$, $F \geq F_n$. As F_n is finite, $F_n \subseteq D$ and $a_1 \oplus a_2 \oplus \cdots \oplus a_n \leq \bigvee F_n = x_{F_n}$, we conclude that u is a compact element of E. Moreover, by [12] for every $x \in E$ we have $x = \bigvee\{u \in E \mid u$ is a finite element$\}$. We conclude that E is compactly generated by finite elements.

(iii) $\Longrightarrow$ (i): This is evident. □

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